# Peculiarities of the Canonical Analysis of the First Order Form of the Einstein-Hilbert Action in Two Dimensions in Terms of the Metric Tensor or the Metric Density

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## Abstract

The peculiarities of doing a canonical analysis of the first order formulation of the Einstein-Hilbert action in terms of either the metric tensor $g^{\alpha\beta}$ or the metric density $h^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$ along with the affine connection are discussed. It is shown that the difference between using $g^{\alpha\beta}$ as opposed to $h^{\alpha\beta}$ appears only in two spacetime dimensions. Despite there being a different number of constraints in these two approaches, both formulations result in there being a local Poisson brackets algebra of constraints with field independent structure constants, closed off shell generators of gauge transformations and off shell invariance of the action. The formulation in terms of the metric tensor is analyzed in detail and compared with earlier results obtained using the metric density. The gauge transformations, obtained from the full set of first class constraints, are different from a diffeomorphism transformation in both cases.

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The Einstein-Hilbert (EH) action in $d$ spacetime dimensions

$$S_d\left(g^{\alpha\beta}\right)=\int d^dx\sqrt{-g}R \tag{1}$$

is known to have very interesting though complicated structure. In two dimensions ($2D$) the action (1) is a total divergence [1] when $R$ is expressed solely in terms of the metric. This does not allow for a study of the basic questions of quantum gravity in the $2D$ limit of (1). Therefore, a variety of alternate $2D$ models of gravity have been developed and investigated. (For a recent review see [2].) Such $2D$ models have an action which is not just a total divergence. In higher dimensions the EH action of (1) is the most interesting model. The canonical treatment of (1) is not only technically complicated, but is peculiar because of the presence of terms with second order derivatives which are not encounted in ordinary gauge theories. In the action of (1), such terms can be put in the form of a total divergence without any additional first order terms [1] but they are needed to retain invariance of the action under general coordinate transformations and must be treated carefully. (For discussion of this problem see [3].)

An efficient and well-known way to avoid this difficulty is to use a first order formulation of (1). In this approach the Dirac procedure [4] can be applied without modification, as in first order formulations of ordinary gauge theories [5]. The first order formulation of (1) was proposed by Einstein [6] (through this is often attributed to Palatini [7]). In the formulation of [6] the Lagrange density takes a simple form

$$L_d=\sqrt{-g}R=h^{\alpha\beta}\left(\Gamma^\lambda_{\alpha\beta,\lambda}-\Gamma^\lambda_{\alpha\lambda,\beta}+\Gamma^\lambda_{\sigma\lambda}\Gamma^\sigma_{\alpha\beta}-\Gamma^\lambda_{\sigma\alpha}\Gamma^\sigma_{\lambda\beta}\right), \tag{2}$$

where the symmetric affine connection $\Gamma^\lambda_{\alpha\beta}$ is considered as an independent field without identifying it with the Christoffel symbol, and $h^{\alpha\beta}=\sqrt{-g}g^{\alpha\beta}$ is the metric density. This formulation was used for the first time by Arnowitt, Deser and Misner (ADM) [8, 9] to study the canonical properties of $S_4$.

In more than two dimensions the second order (1) and first order (2) formulations of EH action are equivalent [6]. In $2D$ they are different; the equation of motion for $\Gamma^\lambda_{\alpha\beta}$ in $2D$ no longer implies that it equals the Christoffel symbol. This was analyzed using the Lagrangian approach in [10]. It is interesting to perform a canonical analysis of the $2D$ EH action in first order form; this is possible as it is no longer a total divergence. In any

case, (2) is formally valid in all dimensions, even though its properties can be quite different in special dimensions. Moreover, the lowest dimensional version of (2) is a $2D$ limit of an action which is equivalent in all higher dimensions to the second order formulation and we would expect that some features would be common to the first order formulation in all dimensions. The simplicity of (2) in $2D$ (e.g., the number of fields is even less than in the first order formulation of the $4D$ Maxwell Lagrangian [5]) allows for a straightforward application of the Dirac procedure without making any *a priori* assumptions or restrictions. The canonical analysis of $L_2(h,\Gamma)$ in (2) using the Dirac procedure, treating the metric density as an independent field, was performed in [11]. The resulting structure is similar to what is encounted in ordinary gauge theories: there is a local algebra of constraints with field independent structure constants, a closed off shell algebra of generators, and a gauge transformation that preserves the exact invariance of $L_2$. The gauge transformation implied by the first class constraints is different from a general coordinate transformation.

A local algebra of constraints was obtained earlier in dilaton gravity [2] but, unlike the algebra of [11], this algebra has field dependent structure constants.

In [11] (as in [6],[8]) $\Gamma^{\sigma}_{\alpha\beta}$ and the metric density $h^{\alpha\beta}$ were used as an independent set of variables. Is it possible that all the desirable canonical properties obtained in [11] are just an artifact of using $h^{\alpha\beta}$ instead of $g^{\alpha\beta}$? This needs to be investigated, as the functional Jacobian $\frac{\delta h^{\alpha\beta}}{\delta g^{\mu\nu}}$, corresponding to a change of variables $h^{\alpha\beta}=\sqrt{-g}g^{\alpha\beta}$, is field dependent in $d>2$ and is singular in $2D$, since in $2D$ the components of $h^{\alpha\beta}$ are not independent, being restricted by the condition $\det\left(h^{\alpha\beta}\right)=-1$. The main goal of this letter is to trace the difference in the canonical analysis when using $g^{\alpha\beta}$ in place of $h^{\alpha\beta}$ and to demonstrate that, despite there being of quite a different constraint structure when using $g^{\alpha\beta}$, all canonical properties found in [11] remain intact and they are not just an artifact of using $h^{\alpha\beta}$.

Straightforward application of the Dirac procedure in [11] automatically gave a local algebra of constraints with field independent structure constants. To preserve these properties and to also have off shell closure of the algebra of generators and exact invariance of the Lagrangian, it was necessary to choose a simple linear transformation of the affine connections as dynamical variables. These transformations were found in component form in [11] but they can be recast into covariant form:

$$\xi^{\lambda}_{\alpha\beta}=\Gamma^{\lambda}_{\alpha\beta}-\frac{1}{2}\left(\delta^{\lambda}_{\alpha}\Gamma^{\sigma}_{\beta\sigma}+\delta^{\lambda}_{\beta}\Gamma^{\sigma}_{\alpha\sigma}\right), \tag{3}$$

so that

$$\Gamma^{\lambda}_{\alpha\beta} = \xi^{\lambda}_{\alpha\beta} - \frac{1}{d-1}\left(\delta^{\lambda}_{\alpha}\xi^{\sigma}_{\beta\sigma} + \delta^{\lambda}_{\beta}\xi^{\sigma}_{\alpha\sigma}\right). \tag{4}$$

Upon substitution into (2) we obtain

$$\tilde{L}_d\left(g,\xi\right) = h^{\alpha\beta}\left(\xi^{\lambda}_{\alpha\beta,\lambda} - \xi^{\lambda}_{\alpha\sigma}\xi^{\sigma}_{\beta\lambda} + \frac{1}{d-1}\xi^{\lambda}_{\alpha\lambda}\xi^{\sigma}_{\beta\sigma}\right). \tag{5}$$

Equation (5) provides an alternative first order formulation of the EH action, as upon substitution of the solution of $\frac{\delta\tilde{L}_d}{\delta\xi^{\lambda}_{\alpha\beta}} = 0$ into $\frac{\delta\tilde{L}_d}{\delta g^{\alpha\beta}} = 0$ one can obtain the usual Einstein field equations without any references to $\Gamma^{\lambda}_{\alpha\beta}$; this calculation is actually simplier than when done using $\Gamma^{\lambda}_{\alpha\beta}$. The main advantage of (5) is the “diagonal” form of the derivative part of the EH action. This is especially well suited for a canonical analysis as the equation

$$h^{\alpha\beta}\xi^{\lambda}_{\alpha\beta,\lambda} = h^{\alpha\beta}\dot{\xi}^{0}_{\alpha\beta} + h^{\alpha\beta}\xi^{k}_{\alpha\beta,k} \tag{6}$$

shows that there is a simple separation of the components of $\xi^{\lambda}_{\alpha\beta}$ into those which are dynamical ( $\xi^{0}_{\alpha\beta}$) and those which are non-dynamical ($\xi^{k}_{\alpha\beta}$). (Latin indices indicate spatial components.) When using the variables $h$ and $\Gamma$, the decomposition (6) is not as simple, since some components of $\Gamma$ enter $L_d$ with both spatial and temporal derivatives, making a straightforward Dirac analysis more difficult.

A full Dirac analysis of $\tilde{L}_d$ for $d > 2$ is beyond the scope of this letter; we shall discuss only the first stage of the Dirac procedure and the question of whether a formulation using the metric tensor is equivalent to one using the metric tensor density. First introducing momenta conjugate to all fields

$$\pi_{\alpha\beta}\left(g^{\alpha\beta}\right), \Pi^{\alpha\beta}_{0}\left(\xi^{0}_{\alpha\beta}\right), \Pi^{\alpha\beta}_{k}\left(\xi^{k}_{\alpha\beta}\right) \tag{7}$$

and using (6) we immediately obtain the $\frac{1}{2}d\left(d+1\right)^{2}$ primary constraints

$$\pi_{\alpha\beta} \approx 0, \Pi^{\alpha\beta}_{k} \approx 0, \Pi^{\alpha\beta}_{0} - \sqrt{-g}g^{\alpha\beta} \approx 0 \tag{8}$$

If the $d\left(d+1\right)$ by $d\left(d+1\right)$ matrix

$$\tilde{M}_d = \left(\left\{\phi, \tilde{\phi}\right\}\right) \tag{9}$$

built from the non-zero Poisson brackets (PB) among the primary constraints $\phi, \tilde{\phi} \in \left(\pi_{\alpha\beta}, \Pi_0^{\alpha\beta} - \sqrt{-g}g^{\alpha\beta}\right)$ is invertible, these constraints are all second class. If the rank of the matrix (9) is $r$, then there are $d\left(d+1\right) - r$ first class constraints. Moreover, all these constraints are constraints of a special form in which one constraint is $\pi_{\alpha\beta} \approx 0$ and the other has $g^{\alpha\beta}$ as a function of the other dynamical variables. (See [4] and for a more detailed and general discussion [12].) For such constraints, if $\det \tilde{M}_d \neq 0$ we can eliminate the momenta $\pi_{\alpha\beta}$ by setting them equal to zero and then solving for $g^{\alpha\beta} = g^{\alpha\beta}\left(\Pi_0^{\gamma\sigma}\right)$ and subsequently substituting this expression into the Hamiltonian and all the remaining constraints. (This is the so-called Dirac or Hamiltonian reduction in its simplest form since in this case the Dirac brackets are equivalent to PB for the remaining variables.) For $\tilde{L}_d$ it is not even necessary to solve any equation for $g^{\alpha\beta}$ as it enters the Hamiltonian in the combination $\sqrt{-g}g^{\alpha\beta}$ which is what is present in the second class primary constraints and the solutions for such combinations are given immediately. Once the canonical analysis gives the gauge transformation for $\Pi_0^{\alpha\beta}$, the equality $\Pi_0^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$ shows how $g^{\alpha\beta}$ itself transforms under a gauge transformation.

In $2D$ (and only in $2D$) the matrix (9) is singular when the metric is used as an independent field and consequently in $2D$, formulations in terms of $g^{\alpha\beta}$ and $h^{\alpha\beta}$ are distinct; we cannot determine the gauge transformation of $g^{\alpha\beta}$ from the gauge transformation of $h^{\alpha\beta}$ found in [11]. In $2D$ the rank of the $6 \times 6$ matrix $\tilde{M}_2$ is four and thus only two pairs of constraints constitute a second class subset of constraints that have a special form that allows two variables to be eliminated by using Dirac reduction. (When the $2D$ EH action is formulated in terms of $h^{\alpha\beta}$, the rank of $\tilde{M}_2$ is six and hence there are three such pairs of second class constraints [11].)

The Lagrangian density when written in component form is given by

$$\tilde{L}_2\left(g,\xi\right) = h^{11}\dot{\xi}_{11}^0 + 2h^{01}\dot{\xi}_{01}^0 + h^{00}\dot{\xi}_{00}^0 - H$$

where

$$H = \xi_{11}^1\left(h_{,1}^{11} - 2h^{11}\xi_{01}^0 - 2h^{01}\xi_{00}^0\right)$$
$$+ 2\xi_{01}^1\left(h_{,1}^{01} + h^{11}\xi_{11}^0 - h^{00}\xi_{00}^0\right) + \xi_{00}^1\left(h_{,1}^{00} + 2h^{01}\xi_{11}^0 + 2h^{00}\xi_{01}^0\right). \tag{10}$$

Note, that in this expression $h^{\alpha\beta}$ is just a short form for $\sqrt{-g}g^{\alpha\beta}$ (and it is not treated as an independent variable) and integration by parts has been performed in the spatial

derivatives. Introducing the momenta (7) conjugate to all fields, we obtain nine primary constraints

$$\pi_{00} \approx 0, \pi_{01} \approx 0, \pi_{11} \approx 0, \tag{11}$$

$$\Pi_1^{11} \approx 0, \Pi_1^{01} \approx 0, \Pi_1^{00} \approx 0, \tag{12}$$

$$\Pi_0^{11} - \sqrt{-g}g^{11} \approx 0, \Pi_0^{01} - \sqrt{-g}g^{01} \approx 0, \Pi_0^{00} - \sqrt{-g}g^{00} \approx 0. \tag{13}$$

We employ the standard fundamental PB

$$\left\{g^{\alpha\beta}, \pi_{\mu\nu}\right\} = \Delta_{\mu\nu}^{\alpha\beta}, \left\{\Gamma_{\alpha\beta}^{\lambda}, \Pi_{\sigma}^{\mu\nu}\right\} = \delta_{\sigma}^{\lambda}\Delta_{\alpha\beta}^{\mu\nu}, \tag{14}$$

where $\Delta_{\mu\nu}^{\alpha\beta} = \frac{1}{2}\left(\delta_\mu^\alpha \delta_\nu^\beta + \delta_\mu^\beta \delta_\nu^\alpha\right)$.

The matrix of PB among the primary constraints (11,13) has rank four and so two pairs of constraints constitute a second class subset. We pick the following pairs

$$\pi_{00} \approx 0, \Pi_0^{00} - \sqrt{-g}g^{00} \approx 0; \pi_{01} \approx 0, \Pi_0^{01} - \sqrt{-g}g^{01} \approx 0. \tag{15}$$

From these we can eliminate $\pi_{00}$, $g^{00}$, $\pi^{01}$ and $g^{01}$ using the strong equations [4]

$$\pi_{00} = 0, g^{00} = g^{11}\frac{\Pi_0^{00}\Pi_0^{00}}{\Pi_0^{01}\Pi_0^{01} - 1}; \pi_{01} = 0, g^{01} = g^{11}\frac{\Pi_0^{00}\Pi_0^{01}}{\Pi_0^{01}\Pi_0^{01} - 1}. \tag{16}$$

To eliminate $g^{01}$ and $g^{00}$ in the Hamiltonian, we actually need only the combinations

$$h^{01} \equiv \sqrt{-g}g^{01} = \Pi_0^{01}, h^{00} \equiv \sqrt{-g}g^{00} = \Pi_0^{00}, \tag{17}$$

For the first constraint of (13) and the Hamiltonian we also need the expression for $\sqrt{-g}g^{11}$, which upon using (16) equals

$$h^{11} \equiv \sqrt{-g}g^{11} = \frac{1}{\Pi_0^{00}}\left(\Pi_0^{01}\Pi_0^{01} - 1\right). \tag{18}$$

This also follows from equality

$$h^{00}h^{11} - h^{01}h^{01} = -\det\left(g_{\alpha\beta}\right)\left(g^{00}g^{11} - g^{01}g^{01}\right) = -1. \tag{19}$$

After the first stage of the Dirac reduction (substitution of (16) into both the constraints and the Hamiltonian), we are left with five primary constraints and we have eliminated two canonical pairs of variables (corresponding to the four primary second class constraints). These five constraints are

$$\Pi_k^{\alpha\beta} \approx 0, \pi_{11} \approx 0, \Pi_0^{11} - \frac{1}{\Pi_0^{00}}\left(\Pi_0^{01}\Pi_0^{01} - 1\right) \approx 0, \tag{20}$$

and the Hamiltonian is

$$H_c = -\xi_{11}^1 \chi_1^{11} - 2\xi_{01}^1 \chi_1^{01} - \xi_{00}^1 \chi_1^{00} \tag{21}$$

where

$$\chi_1^{11} = -\left(\left(\frac{1}{\Pi_0^{00}}\left(\Pi_0^{01}\Pi_0^{01} - 1\right)\right)_{,1} - \frac{2}{\Pi_0^{00}}\left(\Pi_0^{01}\Pi_0^{01} - 1\right)\xi_{01}^0 - 2\Pi_0^{01}\xi_{00}^0\right),$$

$$\chi_1^{01} = -\left(\Pi_{0,1}^{01} + \frac{1}{\Pi_0^{00}}\left(\Pi_0^{01}\Pi_0^{01} - 1\right)\xi_{11}^0 - \Pi_0^{00}\xi_{00}^0\right),$$

$$\chi_1^{00} = -\left(\Pi_{0,1}^{00} + 2\Pi_0^{00}\xi_{01}^0 + 2\Pi_0^{01}\xi_{11}^0\right).$$

Note, that the canonical pair $(g^{11}, \pi_{11})$ is not explicitly present in $H_c$ even though it was not eliminated by the process of Dirac reduction. The role of this "hidden variable" becomes apparent later when the gauge transformation of $g^{\alpha\beta}$ is calculated. Also, if we include a cosmological term linear in $\sqrt{-g}$ in $H_c$, substitution of (16) gives $\sqrt{-g} = \frac{\Pi_0^{01}\Pi_0^{01}-1}{g^{11}\Pi_0^{00}}$ and $H_c$ would then depend explicitly on $g^{11}$ affecting the constraint structure of the model.

All the primary constraints of (20) have vanishing PB among themselves so we go to the next step of the Dirac procedure and consider the persistence of the primary constraints in time. Three of the five primary first class constraints have a non-zero PB with the Hamiltonian, thus producing the secondary constraints

$$\dot{\Pi}_1^{11} = \left\{\Pi_1^{11}, H\right\} = \chi_1^{11}, \dot{\Pi}_1^{01} = \left\{\Pi_1^{01}, H\right\} = \chi_1^{01}, \dot{\Pi}_1^{00} = \left\{\Pi_1^{00}, H\right\} = \chi_1^{00} \tag{22}$$

The constraints (22) have the same algebraic structure as the secondary constraints arising in the $h^{\alpha\beta}$ formulation [11], that is, there is a local algebra of PB with field independent structure constants

$$\left\{\chi_1^{01}, \chi_1^{00}\right\} = \chi_1^{00}, \left\{\chi_1^{01}, \chi_1^{11}\right\} = -\chi_1^{11}, \left\{\chi_1^{11}, \chi_1^{00}\right\} = 2\chi_1^{01}. \tag{23}$$

The Hamiltonian of (21) is a linear combination of these secondary constraints and there are thus no tertiary constraints. (This is not the case when $d > 2$.) The only non zero PB are those given by (23) since all secondary constraints have a vanishing PB with all primary constraints. We have eight first class constraints for the seven pairs of canonical variables remaining after reduction. This seems to give the unphysical result that there are a negative number of degrees of freedom as in the $2D$ gravity models considered in [13]. However, when counting degrees of freedom, we must only include the independent constraints. For the secondary constraints, the following relationship holds

$$\chi_1^{11} - \frac{2\Pi_0^{01}}{\Pi_0^{00}}\chi_0^{01} - \frac{1}{\Pi_0^{00}\Pi_0^{00}}\left(\Pi_0^{01}\Pi_0^{01} - 1\right)\chi_1^{01} = 0. \tag{24}$$

This reduces the number of independent constraints to seven, which implies there are zero degrees of freedom, as expected.

The presence of five primary constraints indicates that the group of gauge transformations has five parameters. Using the approach of Castellani [14], we will recover the gauge transformation of all the initial fields.

The generator $G$ of the gauge transformation is found by first setting $G^b = C_P^b$ for the primary constraints that do not produce secondary first class constraints (i.e. $\left\{C_P^b, H\right\} = 0$) and $G_{(1)}^a = C_P^a$ for primary constraints that produce secondary constraints (i.e $\{C_P^a, H\} \neq 0$). In this case $C_P^b \in \left(\pi_{11}, \Pi_0^{11} - \frac{1}{\Pi_0^{00}}\left(\Pi_0^{01}\Pi_0^{01} - 1\right)\right)$ and $C_P^a \in \left(\Pi_1^{11}, \Pi_1^{01}, \Pi_1^{00}\right)$. We then introduce $G_{(0)}^a(x) = -\{C_P^a, H\}(x) + \int dy\ \alpha_c^a(x,y)\, C_P^c(y)$ where the functions $\alpha_c^a(x,y)$ are found by requiring that $\left\{G_{(0)}^a, H_c\right\} = 0$. (In our model, not all of the functions $\{C_P^a, H\}(x)$ are independent, which is a situation distinct from what was considered in [14]. We do however construct a generator of a gauge transformation which leaves the action invariant even off shell.) The full generator of gauge transformations is then given by

$$G\left(\varepsilon^b; \varepsilon^a, \dot{\varepsilon}^a\right) = \int dx \left(\varepsilon^b(x)\, G^b(x) + \varepsilon^a(x)\, G_{(0)}^a(x) + \dot{\varepsilon}^a(x)\, G_{(1)}^a(x)\right).$$

In our case this leads to the following expression

$$\begin{aligned} G(\varepsilon) = \int dx \Bigg[ & \varepsilon^{11}\pi_{11} + \varepsilon_{11}\left(\Pi_0^{11} - \frac{1}{\Pi_0^{00}}\left(\Pi_0^{01}\Pi_0^{01} - 1\right)\right) \\ & + \varepsilon\left(-\chi_1^{01} - \xi_{00}^1\Pi_1^{00} + \xi_{11}^1\Pi_1^{11}\right) + \dot{\varepsilon}\Pi_1^{01} \\ & + \varepsilon_1\left(-\chi_1^{11} - 2\xi_{01}^1\Pi_1^{11} - 2\xi_{00}^1\Pi_1^{01}\right) + \dot{\varepsilon}_1\Pi_1^{11} + \varepsilon^1\left(-\chi_1^{00} + 2\xi_1^{11}\Pi_1^{01} + 2\xi_{01}^1\Pi_1^{00}\right) + \dot{\varepsilon}^1\Pi_1^{00}\Bigg]. \end{aligned} \tag{25}$$

The PB of the generators (25) has an algebra similar to what appears in [11] when $h^{\alpha\beta}$ is treated as an independent field

$$\{G(\varepsilon), G(\eta)\} = G\left(\tau^c = C^{cab}\varepsilon^a\eta^b)\right) \tag{26}$$

where $\varepsilon^a = (\varepsilon^1(\varepsilon), \varepsilon^2(\varepsilon_1), \varepsilon^3(\varepsilon^1), \varepsilon^4(\varepsilon_{11}), \varepsilon^5(\varepsilon^{11}))$ and the only non-zero structure constants $C^{cab}$ are $C^{132} = 2 = -C^{123}$, $C^{212} = 1 = -C^{221}$, $C^{331} = 1 = -C^{313}$. This reflects the structure of algebra of PB among all first class constraints.

Using (25) we can determing the gauge transformation of the seven pairs of phase space variables remaining after the Dirac reduction by using the equation $\delta field = \{field, G(\varepsilon)\}$.

The transformations of the fields $\Pi_1^{\alpha\beta}$ and $\xi_{\alpha\beta}^1$ are the same as in [11] where $h^{\alpha\beta}$ is treated as an independent field

$$\delta\Pi_1^{01} = \varepsilon_1\Pi_1^{11} - \varepsilon^1\Pi_1^{00}, \delta\Pi_1^{00} = \varepsilon\Pi_1^{00} + 2\varepsilon_1\Pi_1^{01}, \delta\Pi_1^{11} = -\varepsilon\Pi_1^{11} - 2\varepsilon^1\Pi_1^{01}, \tag{27}$$

$$\delta\xi_{01}^1 = \frac{1}{2}\dot{\varepsilon} + \varepsilon^1\xi_{11}^1 - \varepsilon_1\xi_{00}^1, \delta\xi_{00}^1 = \dot{\varepsilon}^1 - \varepsilon\xi_{00}^1 + 2\varepsilon^1\xi_{01}^1,$$
$$\delta\xi_{11}^1 = \dot{\varepsilon}_1 + \varepsilon\xi_{11}^1 - 2\varepsilon_1\xi_{01}^1. \tag{28}$$

There are slightly modified transformations for $\Pi_0^{\alpha\beta}$

$$\delta\Pi_0^{00} = \varepsilon\Pi_0^{00} + 2\varepsilon_1\Pi_0^{01}, \delta\Pi_0^{01} = \varepsilon_1\left(\frac{1}{\Pi_0^{00}}\left(\Pi_0^{01}\Pi_0^{01} - 1\right)\right) - \varepsilon^1\Pi_0^{00},$$
$$\delta\Pi_0^{11} = -\varepsilon\left(\frac{1}{\Pi_0^{00}}\left(\Pi_0^{01}\Pi_0^{01} - 1\right)\right) - 2\varepsilon^1\Pi_0^{01}, \tag{29}$$

while those for $\xi_{\alpha\beta}^0$ are quite distinct

$$\delta\xi_{11}^0 = \varepsilon_{11}, \tag{30}$$

$$\delta\xi_{00}^0 = -\varepsilon_{,1}^1 - \varepsilon\xi_{00}^0 + 2\varepsilon^1\xi_{01}^0 + \frac{1}{\Pi_0^{00}\Pi_0^{00}}\left(\Pi_0^{01}\Pi_0^{01} - 1\right)\left(\varepsilon_{11} + \varepsilon_{1,1} + 2\varepsilon_1\xi_{01}^0 - \varepsilon\xi_{11}^0\right), \tag{31}$$

$$\delta\xi_{01}^0 = -\frac{1}{2}\varepsilon_{,1} + \varepsilon^1\xi_{11}^0 - \varepsilon_1\xi_{00}^0 - \frac{1}{\Pi_0^{00}}\Pi_0^{01}\left(\varepsilon_{11} + \varepsilon_{1,1} + 2\varepsilon_1\xi_{01}^0 - \varepsilon\xi_{11}^0\right). \tag{32}$$

There is also a new pair of transformations

$$\delta\pi_{11} = 0, \delta g^{11} = \varepsilon^{11}, \tag{33}$$

which reflects the disappearence, noted above, of $g^{11}$ from $H_c$ once the Dirac reduction has been performed. We obtain the transformation of $h^{\alpha\beta}$ by using the first two of equations (29) and strong equalities (16,18)

$$\delta h^{00} = \varepsilon h^{00} + 2\varepsilon_1 h^{01}, \delta h^{01} = \varepsilon_1 h^{11} - \varepsilon^1 h^{00}. \tag{34}$$

Note, that the third equation of (29) gives $\delta \Pi_0^{11} = -\varepsilon h^{11} - 2\varepsilon^1 h^{01}$ but we cannot use this to find $\delta h^{11}$ as $\Pi_0^{11} \approx h^{11}$ only weakly. To determine the variation $\delta h^{11}$ we have to use (19) or alternatively (20) and then use (34) to obtain

$$\delta h^{11} = -\varepsilon h^{11} - 2\varepsilon^1 h^{01}. \tag{35}$$

The transformation of $h^{\alpha\beta}$ we have found is the same as in [11] where $h^{\alpha\beta}$ is treated as being independent.

Finally, the transformations for $g^{00}$ and $g^{01}$ can be found using the strong equalities (16,17) and the transformations (33,34)

$$\delta g^{00} = \varepsilon^{11}\frac{g^{00}}{g^{11}} + \varepsilon 2g^{00} + \varepsilon^1 \frac{2g^{00}g^{01}}{g^{11}} + \varepsilon_1 2g^{01}, \tag{36}$$

$$\delta g^{01} = \varepsilon^{11}\frac{g^{01}}{g^{11}} + \varepsilon g^{01} + \varepsilon^1 \frac{2g^{01}g^{01}}{g^{11}} + \varepsilon_1 g^{11} - \varepsilon^1 g^{00}. \tag{37}$$

Equations (28,30-33,36,37) give the transformations of all fields in the original action (5). (In (31,32) we have had to express $\Pi_0^{00}$ and $\Pi_0^{01}$ in terms of $h^{\alpha\beta}\left(g^{\alpha\beta}\right)$ using equalities (16,18).) To check the invariance of the Lagrangian we can use (34,35), as $g^{\alpha\beta}$ enters $\tilde{L}_2$ only through $h^{\alpha\beta}$. In this case the fifth parameter of the gauge transformations $\varepsilon^{11}$ becomes "hidden" (as it enters only in transformations of $g^{\alpha\beta}$). However, as we have five primary constraints, there should be five parameters. Without this parameter it is impossible to obtain the transformations of all the components of $g^{\alpha\beta}$. All these peculiarities make the $2D$ action a very interesting model from point of view of constraint dynamics.

The variation of $\tilde{L}_2$ in (5) gives

$$\delta\tilde{L}_2 = 2\left[\left(h^{11} - \frac{h^{01}h^{01}}{h^{00}}\right)\left(\varepsilon_{11} + \varepsilon_{1,1} + 2\varepsilon_1\xi^0_{01} - \varepsilon\xi^0_{11}\right)\right]_{,0} \tag{38}$$

and the action is invariant provided this total derivative can be neglected.

It is not clear why the gauge transfromation that leaves $\tilde{L}_2$ invariant in the $h$ formulation only leaves $\tilde{L}_2$ up to a total time derivative in the $g$ formulation. Possibly, the linear dependence of secondary constraints in (24) has to be taken into account when applying the Castellani procedure.

One can also find the transformation of the affine connection by using (4).

## CONCLUSION

We have demonstrated that the canonical procedure applied to the EH action in $2D$ with $g^{\alpha\beta}$ being independent leads to the same canonical structure as in the approach where the metric density $h^{\alpha\beta}$ is an independent field. It also leads to a gauge invariance which is different from a diffeomorphism. This is despite having a different number of constraints in the two formulations.

Based on these results in $2D$ and some preliminary investigations of the higher dimensional EH action, we are tempted to conclude that these canonical properties should be preserved for gravity in all dimensions. We think that it is highly unlikely that the theory (first order formulation of EH), which behaves as a local field theory in $2D$, is not a local field theory when $d > 2$. Beyond two dimensions, straightforward application of the Dirac approach is complicated (even with the simplification of using $\xi^\lambda_{\alpha\beta}$ in place of $\Gamma^\lambda_{\alpha\beta}$ ) by the fact that the Hamiltonian is no longer a linear combination of secondary constraints as in (21) and furthermore at least tertiary constraints appear.

If $d > 2$, all secondary constraints coresponding to the momenta $\Pi^{nk}_m$ and $\Pi^{0k}_n$ (7) (except the momentum $\Pi^{0k}_k$) constitute a special set of second class constraints which can be eliminated. Thus, if $d > 2$, there are $d$ primary and $d$ secondary constraints and they produce $d$ tertiary constraints. Although this work is in progress, we can make the argument that if the $d$ tertiary constraints are also first class and the Dirac procedure does not lead to any further constraints, than we have $3d$ first class constraints. The fields left after using the second class constraints to eliminate some variables through the Dirac reduction are the $\frac{1}{2}d\,(d+1)$

components of the symmetric second rank tensor ($\xi^0_{\alpha\beta}$) plus the $(d-1)$ components of $\xi^0_{0k}$ plus the one component $\xi^l_{0l}$; this gives the total number of variables after the Dirac reduction to be $\frac{1}{2}d\,(d+3)$. Subtracting the number of first class constraints $(3d)$ leaves us with $\frac{1}{2}d\,(d-3)$ degrees of freedom. This is the number of degrees of freedom present in a spin two gauge field in any dimension. Of course, this scenario with $3d$ first class constraints is not the only possibility that gives a correct expression for this number of degrees of freedom, but it does illustrate that the presence of tertiary constraints does not contradict having the anticipated number of degrees of freedom.

The ADM [8, 9] analysis of first order formulation of the EH action ends with secondary constraints where we, using the Dirac reduction, have at least tertiary constraints. The difference between the two approaches is based on the fact that in the ADM procedure not only second class, but also first class constraints were solved during the preliminary Lagrangian reduction of the action. This is because all time independent equations of motion are used to eliminate some variables without distinguishing whether these equations correspond to first class or second class constraints. (See a very clear exposition of this procedure in Appendix I of the review article [15].) Solutions of the 30 constraint equations (the number of field equations without time derivatives of fields in $4D$) after substitution back into the original Lagrangian leads to a disappearance of 34 variables (a clear indication that four first class constraints have been solved) and the reduced action hence has only 16 variables (see Eq.(4.1) of [9]). In our approach in which only the second class constraints are eliminated, after the Dirac reduction 24 variables are left. A more detailed discussion of this point will be published elsewhere.

An important question for the higher dimensional EH action is whether it is possible to preserve all canonical properties present in the $2D$ case by following the standard Dirac procedure without any *a priori* assumptions and restrictions, and to determine what gauge transformation this procedure produces. (It is possible that a diffeomorphism is not *the only* symmetry of $d$ dimensional EH action; we have seen that in fact an alternate symmetry occurs in $2D$.) If the final algebraic structure of PB of first class constraints is local, then a viable approach to quantizing higher dimensional gravity may exist.

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